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BRIEF COMMUNICATION

THERMOCAPILLARY AND BUOYANT BUBBLE MOTION WITH VARIABLE VISCOSITY

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1. INTRODUCTION

The purpose of this paper is to analyze the combined effect of thermocapillary migration and buoyancy induced motion of a gas bubble in an immiscible liquid in which a uniform temperature gradient exists. It is assumed that the Reynolds number of the motion is large and the bubble is undeformed. The directions of the gravity vector and the temperature gradient are assumed to be parallel or anti-parallel.

Thermocapillary migration of bubbles and drops is important in microgravity and has been reviewed by Subramanian (1992). Recent results of a spaceflight experiment have been reported by Balasubramaniam *et al.* (1996). Thermocapillary migration and its interaction with buoyancy are also important under normal gravity in preparation for space experiments (Srividya 1993) and for special applications such as the processing of bi-metallic composites where gravitational sedimentation has to be reduced (Zhang *et al.* 1993; Prinz and Romero 1992).

The thermocapillary motion of a bubble for large values of the Reynolds number in the absence of buoyancy has been analyzed both in the limit of zero Marangoni numbers (Crespo and Manuel 1983; Balasubramaniam and Chai 1987) and large Marangoni numbers (Crespo and Jimenez-Fernandez 1991; Balasubramaniam and Subramanian 1996). In all these studies the liquid viscosity was assumed to be a constant, independent of the local temperature. The fundamental conclusion of the studies at large Marangoni numbers is that a suitably scaled result for the bubble velocity tends to a non-zero asymptotic value. The goal of the present work is first to determine the effects of buoyancy on this asymptotic limit when the Peclet number is large and second to address the issue of temperature dependence of the viscosity. Both these questions are examined when the Reynolds number is large as this will permit simplification in the description of the flow field.

2. PROBLEM FORMULATION

The following assumptions are made: the flow is governed by the incompressible Navier–Stokes equations, with surface tension and viscosity being linearly dependent on the temperature; the Reynolds number of the flow is large, thus to leading order the flow is given by potential flow everywhere (including the momentum boundary layer, Moore 1963; Balasubramaniam and Subramanian 1996); the bubble is spherical in shape, that is, the Weber number is sufficiently small such that shape deformations are negligible; the Peclet number of the motion is large; the steady gravity vector and the temperature gradient are either parallel or anti-parallel so that before the bubble is introduced the liquid is quiescent—in the unstably stratified case it is assumed that the critical Rayleigh number for Rayleigh–Benard instability is not exceeded. The motion due to natural convection is not considered.

Note that for large Reynolds numbers the leading order flow field (in a large Re asymptotic expansion) is given by potential flow regardless of whether viscosity is constant or variable. It is this crucial fact that enables the results given by Balasubramaniam and Subramanian (1996) to be extended when viscosity variations are included.

We will assume that thermocapillarity is predominant over buoyancy and the bubble moves in the direction of the temperature gradient. As will be shown later, it is trivial to write the results for a case when buoyancy is dominant and the bubble moves in a direction opposite to the temperature gradient. In a reference frame moving with the bubble the scaled velocity field given by potential flow is

$$u = -v_\infty \cos\theta\left(1 - \frac{1}{r^3}\right), \qquad v = v_\infty \sin\theta\left(1 + \frac{1}{2r^3}\right) \tag{1}$$

Here u, v are the radial and tangential velocities in spherical polar coordinates whose origin moves with the bubble. v_∞ is the bubble velocity that is to be determined. The Reynolds and Peclet numbers are defined as $\mathrm{Re} = V_R R_1/\nu$ and $Pe = V_R R_1/\alpha$ where ν, α are the kinematic viscosity and thermal diffusivity of the liquid. The thermocapillary velocity scale is $V_R = (-\sigma_T)AR_1/\mu_0$ where σ_T is the rate of change of surface tension with temperature that is assumed to be a negative constant, A is the magnitude of the applied temperature gradient, R_1 is the radius of the bubble and μ_0 is the viscosity of the undisturbed liquid at the same location as the center of the bubble. For purely thermocapillary flow, the Peclet number is called the Marangoni number.

The energy equation for the scaled transformed temperature $T = \bar{T} - AV_R v_\infty \tau / AR_1$ where $\bar{T}$ is the physical temperature and τ denotes time is

$$v_\infty + \mathbf{u}\cdot\nabla T = \frac{1}{\mathrm{Pe}}\nabla^2 T \qquad T \rightarrow r\cos\theta \text{ as } r \rightarrow \infty \qquad \frac{\partial T}{\partial r} = 0 \text{ at } r = 1 \tag{2}$$

In the energy equation we have assumed a quasi-steady distribution of temperature. The steady state assumption is strictly valid only for constant v_∞ which implies a constant viscosity.

The bubble velocity is determined from the conservation of mechanical energy: the work done by the thermocapillary shear stress and the buoyancy force is equal to the rate at which kinetic energy is dissipated by viscosity. The scaled mechanical energy conservation equation can be written as

$$-\int_0^\pi \left(v\frac{\partial T}{\partial\theta}\right)_{r=1} \sin\theta \, \mathrm{d}\theta \pm \frac{2}{3}\mathrm{Bo}v_\infty = \int_1^\infty \int_0^\pi \frac{\mu(T)}{\mu_0} \nabla\cdot\nabla(\mathbf{u}\cdot\mathbf{u})r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \tag{3}$$

where $\mathrm{Bo} = (\rho_\mathrm{L} - \rho_\mathrm{G})gR_1/((-\sigma_T)A)$ is a dynamic Bond number. The terms on the left hand side are the contributions to the rate of work done by the thermocapillary stress and buoyancy forces respectively. The plus sign for the rate of work done by the buoyancy force is used when the bubble moves in the direction of the buoyancy force acting on it; the minus sign is used if it moves in the opposite direction. The right hand side of [3] represents the rate of energy dissipation by viscous forces for the potential flow field. In deriving [3] the rate of change of kinetic energy in the liquid is neglected; this can be shown to be valid when $\mathrm{Re}\, \mathrm{d}\mu/\mathrm{d}T\, AR_1/\mu \ll 1$.

When viscosity is constant and gravitational forces are absent, Balasubramaniam and Subramanian (1996) show that the asymptotic solution for large Pe (or Ma) is

$$v_\infty = v_{\infty 0} + o(\epsilon) \tag{4}$$

$$T = r\cos\theta + \int_r^\infty \frac{1}{(\tilde{r}^3 - 1)} \frac{\left(\dfrac{3\psi_0}{v_{\infty 0}(\tilde{r}^2 - \frac{1}{\tilde{r}})} - 1\right)}{\left(1 - \frac{2\psi_o}{v_{\infty 0}(\tilde{r}^2 - \frac{1}{\tilde{r}})}\right)^{1/2}} d\tilde{r} \tag{5}$$

$$t = \frac{1}{3}\ln\epsilon + 1 + \frac{\pi}{6\sqrt{3}} - \frac{1}{6}\ln(27v_{\infty 0}^2) + \frac{2}{3}\ln\left(\frac{1+\cos\theta}{\sin\theta}\right) + \frac{1}{6}\ln\xi + \frac{2}{3}F(\zeta) \quad [6]$$

where t and T denote the temperature field in the thermal boundary layer and outside it, respectively,

$$\epsilon = 1/\sqrt{\text{Pe}},\ \psi_0 = \frac{v_{\infty 0}}{2}\sin^2\theta\left(r^2 - \frac{1}{r}\right),\ \xi = \frac{v_{\infty 0}}{2}(1-\cos\theta)^2(2+\cos\theta),\ x = \frac{(r-1)}{\epsilon}$$

$$\zeta = \frac{3v_{\infty 0}}{4\sqrt{\xi}}x\sin^2\theta,\ F(\zeta) = \int_0^\zeta D(w)\mathrm{d}w = \int_0^\zeta e^{-w2}\left(\int_0^w e^{z^2}\ \mathrm{d}z\right)\mathrm{d}w$$

($D(x)$ denotes Dawson's function, see Abramowitz and Stegun 1968). The bubble velocity $v_{\infty 0}$ is computed from [3] and [5] to be $v_{\infty 0} = 1/3 - (\ln 3)/8$.

One way to accomodate the variation of viscosity with temperature in the results from a constant property theory is via a quasi-static approach. Basically the reference value for the viscosity μ_0 in the expression for the velocity scale V_R is permitted to depend on the location of the bubble. Recall that μ_0 is the viscosity of the undisturbed liquid in a plane perpendicular to the direction of motion that contains the center of the bubble; in the quasi-static approach this is permitted to be a function of the temperature of the undisturbed liquid. For small ϵ, it is seen from [6] that the temperature on the bubble surface scales as ln ϵ, while the temperature gradient itself is $O(1)$. Thus the fluid near the bubble is cooler than further away, in a plane perpendicular to the direction of motion. Since the viscosity of the liquid increases with decreasing temperature, the liquid is more viscous near the bubble than further away from it in this plane. Thus the actual migration speed of the bubble is expected to decrease compared to a quasi-static prediction that uses the reference viscosity μ_0. Balasubramaniam and Subramanian (1996) explain that the relatively cold condition prevalent at the bubble surface for small ϵ is related to the long transit times, near the front stagnation point, of fluid elements in a thin bundle that surround the front stagnation streamline.

3. RESULTS

We first assume that the viscosity is constant during the motion of the bubble, but the Bond number is not equal to zero. When the Peclet number is large, the temperature field is given by [5] and [6]. What is altered by the buoyancy force is the migration velocity of the bubble. Substituting [1] and [6] into [3] the bubble velocity may be determined to be

$$v_{\infty 0} = \frac{1}{3} - \frac{1}{8}\ln 3 \pm \frac{1}{9}\text{Bo} \quad [7]$$

The direction of motion of the bubble is the direction of the temperature gradient when σ_T is negative. The positive sign in the above formula is used when ρ_L, ρ_G and g are such that buoyancy enhances the motion; the negative sign is used otherwise. Even though the problem for v_∞ is non-linear, [7] reveals that in the limit of large values for Re and Pe the bubble velocity at leading order is a superposition of the thermocapillary and buoyancy induced migration velocities. This linear behavior is unusual and obtains in this limit because the potential velocity field for large Re [1] is coupled to the temperature field solely via the quantity v_∞ that acts as a scale factor; the flow field is otherwise unaltered. Further, when Pe is large, [6] reveals that the temperature gradient at the bubble surface that is responsible for the flow is independent of v_∞. For values of the Reynolds and Peclet numbers that are not large, the linear superposition of the thermocapillary and buoyancy induced migration velocities will not be valid.

The viscosity of the liquid is now permitted to be a linear function of temperature. As the bubble moves toward warmer liquid, the temperature at the reference location for the viscosity changes continually. Thus μ_0 is a time-dependent quantity. The viscosity variation with temperature is expressed as

$$\mu(\bar{T}) = \mu_0(\bar{T}_0) + \frac{d\mu}{dT}(\bar{T} - \bar{T}_0) \quad \text{or} \quad \frac{\mu}{\mu_0} = 1 + \frac{AR_1}{\mu_0}\frac{d\mu}{dT}T \qquad [8]$$

Here $d\mu/dT$ is taken to be a constant and is typically negative for liquids. In reality the viscosity has an exponential dependence on the absolute temperature. However it can be shown that $\nabla\cdot\nabla(\mathbf{u}\cdot\mathbf{u})$ in the integrand in the right hand side of [3] decays as r^{-8}. Thus the principal contribution to the integral occurs in a region close to the bubble. When the viscosity is a non-linear function of the temperature, an average value for $d\mu/dT$ in this region can be calculated and used in [8].

When the transport of energy is assumed to be quasi-static, the temperature field around the bubble is described by [5] and [6]. Substituting [1], [5] and [8] into [3], the scaled rate of energy dissipation by viscous forces may be obtained as

$$\dot{E} = 12\pi v_{\infty 0}^2 + 18\pi K\frac{AR_1}{\mu_0}\frac{d\mu}{dT}v_{\infty 0}^2 \qquad [9]$$

$$K = \int_1^\infty \int_0^\pi \int_r^\infty \frac{\left[\frac{3}{2}\left(r^2 - \frac{1}{r}\right)\sin^2\theta - \left(s^2 - \frac{1}{s}\right)\right]}{\left[\left(s^2 - \frac{1}{s}\right) - \left(r^2 - \frac{1}{r}\right)\sin^2\theta\right]^{1/2}} \frac{\sin\theta(1 + 2\cos^2\theta)}{r^6(s^3 - 1)\left(s^2 - \frac{1}{s}\right)^{1/2}} dr\, d\theta\, ds = -0.326 \qquad [10]$$

The triple integral has been evaluated numerically. The use of the outer temperature field [5] in determining $\dot{E}$ requires some comment. As mentioned before the main contribution to the integral that determines $\dot{E}$ occurs in a region near the bubble where $\nabla\cdot\nabla(\mathbf{u}\cdot\mathbf{u})$ is non-vanishing. This region is expected to be much thicker than the thermal boundary layer which scales as $\text{Pe}^{-1/2}$ in the limit $\text{Pe} \to \infty$. Thus the contribution to the dissipation within the thermal boundary layer appears negligible. However, the outer temperature field T given by [5] contains a logarithmic singularity of the form $(1/3)\ln(r - 1)$ (see Balasubramaniam and Subramanian 1996). While this singularity is integrable in the evaluation of K in [10], such an integrable form of the integrand in K is a consequence of the assumption that the viscosity varies linearly with temperature. If a more general viscosity-temperature relationship such as an exponential variation is used, the use of the outer temperature field alone will not yield a finite value for $\dot{E}$ and the existence of the thermal boundary layer cannot be ignored. The expression for $\dot{E}$ will be integrable if a composite temperature field constructed from [5] and [6] is used in the right hand side of [3].

The temperature on the bubble surface given by [6] is used in the left hand side of [3] to determine the rate at which work is done by the thermocapillary stress. Using the expression for the rate of dissipation of energy given by [9] the migration velocity of the bubble may be determined to be

$$v_{\infty 0} = \frac{\frac{1}{3} - \frac{1}{8}\ln 3 \pm \frac{1}{9}\text{Bo}}{1 + \frac{3}{2}K\frac{AR_1}{\mu_0}\frac{d\mu}{dT}} \qquad [11]$$

As before the plus sign for the buoyancy term is used when buoyancy enhances the motion of the bubble. Consider the thermocapillary motion of a bubble in reduced gravity in a silicone oil with a room temperature viscosity of 10 cp. For this liquid $(1/\mu_0)(d\mu/dT)$ is approximately $-0.02\ \text{K}^{-1}$ and σ_T is approximately -0.06 mN/m. For a bubble size of 10 mm and a temperature gradient of 1.5 K/mm, the Reynolds and Marangoni numbers are approximately 20 and 2000 respectively. [11] predicts that the temperature dependence of viscosity reduces the migration speed of the bubble by about 7.3%. However, Re $(d\mu/dT)AR_1/\mu_0 \ll 1$ is not satisfied in this example and the theory that neglects the rate of change of kinetic energy is not really acceptable.

Equation [11] can be generalized for the dimensional bubble migration velocity vector $\mathbf{V}_{\infty 0}$ as

$$\mathbf{V}_{\infty \mathbf{0}} = \frac{\left(\frac{1}{3} - \frac{1}{8}\ln 3\right)\frac{(-\sigma_T)AR_1}{\mu_0}\frac{\nabla T_\infty}{|\nabla T_\infty|} - \frac{1}{9}\frac{(\rho_L - \rho_G)gR_1^2}{\mu_0}\frac{\boldsymbol{g}}{|\boldsymbol{g}|}}{\left(1 + \frac{3}{2}K\frac{AR_1}{\mu_0}\frac{d\mu}{dT}\frac{\mathbf{V}_{\infty \mathbf{0}} \cdot \nabla T_\infty}{|\mathbf{V}_{\infty \mathbf{0}} \cdot \nabla T_\infty|}\right)} \tag{12}$$

where ∇T_∞ is the applied temperature gradient whose magnitude is A and $\boldsymbol{g}$ is the gravity vector that is either parallel or anti-parallel to the temperature gradient. The variation of viscosity with temperature reduces the bubble speed when it moves in the direction of the temperature gradient and enhances it when it moves in the opposite direction.

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REFERENCES

Abramowitz, M. and Stegun, I. (1968) *Handbook of Mathematical Functions*. Dover, New York.

Balasubramaniam, R. and Chai, A. T. (1987) Thermocapillary migration of droplets: an exact solution for small Marangoni numbers. *J. Colloid Interface Sci.* **119,** 531–538.

Balasubramaniam, R., Lacy, C. E., Wozniak, G. and Subramanian, R. S. (1996) Thermocapillary migration of bubbles and drops at moderate values of the Marangoni number in reduced gravity. *Phys. Fluids* **8,** 872–880.

Balasubramaniam, R. and Subramanian, R. S. (1996) Thermocapillary bubble migration–thermal boundary layers for large Marangoni numbers. *Int. J. Multiphase Flow* **22,** 593–612.

Crespo, A. and Manuel, F. (1983) Bubble motion under reduced gravity. In *Proceedings of the 4th European Symposium on Materials Sciences Under Microgravity*. 45–49. Madrid, Spain.

Crespo, A. and Jimenez-Fernandez, J. (1992) Thermocapillary migration of bubbles at moderately large Reynolds numbers. In *Microgravity Fluid Mechanics*. H. J. Rath, ed. Proc. IUTAM Symposium Bremen 1991. Springer, Berlin.

Moore, D. W. (1963) The boundary layer on a spherical gas bubble. *J. Fluid Mech.* **16,** 161–176.

Prinz., B. and Romero, A. (1992) New casting process for hypermonotectic alloys. In *Proceedings of International Workshop on Systems with a Miscibility Gap*, ed. L. Ratke. Bad Honnef, Germany.

Srividya, C. V. (1993) Migration of Fluorinert FC-75 drops in a silicone oil in a vertical temperature gradient. M.S. thesis. Department of Chemical Engineering, Clarkson University, Potsdam, New York.

Subramanian, R. S. (1992) The motion of bubbles and drops in reduced gravity. In *Transport Processes in Bubbles, Drops and Particles*, ed. R. P. Chhabra and D. DeKee, pp. 1–42.

Zhang, X., Wang, H. and Davis, R. H. (1993) Collective effects of temperature gradients and gravity on droplet coalescence. *Phys. Fluids A* **5,** 1602–1613.